COMBINATORICA 5 (3) (1985) 247—255

A STRONGLY POLYNOMIAL MINIMUM COST CIRCULATION ALGORITHM

ÉVA TARDOS

Received 5 November 1984

A new algorithm is presented for the minimum cost circulation problem. The algorithm is strongly polynomial, that is, the number of arithmetic operations is polynomial in the number of nodes, and is independent of both costs and capacities.

1. Introduction

Edmonds and Karp [7] developed a scaling technique to solve the minimum cost flow problem in polynomial time. The number of arithmetic operations used by their algorithm depends on the size of the capacities. Röck [14] described a variant of their method, where the number of arithmetic operations depends on the size of the costs. Edmonds and Karp [7] write:

A challenging open problem is to give a method for the minimum cost flow problem having a bound of computation which is a polynomial in the number of nodes, and is independent of both costs and capacities.

Here we present such an algorithm. The algorithm consists of phases (at most as many as the number of edges). In each phase one has to solve a minimum cost flow problem with "rounded" costs (that is, the costs will be integers and will have absolute value at most $\lceil |V|\sqrt{|E|}\rceil$, a polynomial in the size of the graph). Since the costs are small integers the known methods (even without the scaling technique, see Edmonds—Karp [7]) give strongly polynomial algorithms for these subproblems.

In the model of computation considered here the input consists of rational numbers. Each number in the data adds one to the input size. Each arithmetic operation (addition, multiplication, division and comparison) is considered as one step. An algorithm which has polynomial running time in this model may turn out to be not polynomial in the usual sence. This happens if the size of the numbers occurring during the algorithm is not polynomially bounded in the size of the input numbers.

AMS subject classification (1980): 68 E 10, 68 C 25.

An algorithm is said to be *strongly polynomial* if
— the number of arithmetic operations is polynomially bounded in the above input size and
— the size of the numbers occurring during the algorithm is polynomially bounded in the size of the input numbers.

(Megiddo 1981 uses the term "genuinely polynomial".) Actually a strongly polynomial algorithm can be interpreted to work on any real numbers (not only on rationals) as well. Given any subfield of the reals, where the above four arithmetic operations can be carried out, a strongly polynomial algorithm works on problems with numbers given from that field. Such a subfield is, for example, the field of numbers $a+b\sqrt{2}$ for a, b rationals. The only apparent danger is that the length of the encoding of the numbers occurring during the algorithm will not be polynomially bounded in the length of the encoding of the input numbers.

As an application let us draw attention to a paper of Anstee [1] where Anstee gives a new polynomial algorithm for solving the minimum cost b-matching problem. The only portion of his algorithm, that is not strongly polynomial is the solution of a minimum cost circulation problem. Combining the present algorithm with the algorithm of Anstee one gets a strongly polynomial algorithm for the minimum cost b-matching problem.

2. Preliminaries

In the following let $D=(V,E)$ denote a digraph (possibly with multiple edges) and let us be given upper and lower capacities and costs $g, f, d\in\mathbf{R}^E$, respectively, on the edges (satisfying $f\leqq g$). The case when f and g may have infinite components will be discussed in part 5.

A vector $x\in\mathbf{R}^E$ is a *circulation* if for each node $v\in V$ we have

$$\sum\left(x(e)\colon\ e=uv\in E\right)-\sum\left(x(e)\colon\ e=vu\in E\right)=0.$$

A circulation is *feasible* if $f\leqq x\leqq g$. The set of feasible circulations is a polytope. It is called the *circulation polytope* and it is denoted by $P(f,g)$. The *circulation problem* is to find a feasible circulation if one exists. It is well-known that the circulation problem can be solved in strongly polynomial time (Dinits [3], Edmonds—Karp [7]). The *cost* of a circulation x is

$$dx=\sum\left(d(e)x(e)\colon\ e\in E\right).$$

The *minimum cost circulation problem* is to find a feasible circulation with minimal cost. Such a circulation is called *d-minimal.*

We call an edge *tight* if $f(e)=g(e)$ and the set of all tight edges is denoted by $T(f,g)$.

To define a face of the circulation polyhedron one has to replace some of the defining inequalities $x(e)\geqq f(e)$, $x(e)\leqq g(e)$ (for $e\in E$) by equalities. Any face of the circulation polytope is obviously a circulation polytope itself. The set of d-minimal circulations forms a face P_{opt} called the *d-minimizer of* P. The algorithm will determine this face.

For a vector $x\in\mathbf{R}^E$ let $x_{\max}=\max(|x(e)|$ for $e\in E)$ and $|x|=(\sum x^2(e)$: $e\in E)^{1/2}$. For a real number λ let $\lceil\lambda\rceil$ denote the smallest integer greater than or equal to λ.

Let us call any vector $\pi\in\mathbf{R}^V$ a *potential.*

Two cost functions d and d' are said to be *(f, g)-equivalent* if there exists a potential π such that for each non-tight edge $e=uv$ we have

$$d'(e) = d(e)+\pi(u)-\pi(v).$$

Lemma 2.1. *Let d and d' be (f, g)-equivalent cost functions. A feasible circulation x is d-minimal if and only if it is d'-minimal.*

Proof. We prove that $dx-d'x$ does not depend on the feasible circulation x. Indeed,

$$d'x-dx = \sum\big(x(e)(\pi(u)-\pi(v)):\ e = uv\in E\big)+$$
$$+\sum\big(x(e)(d'(e)-d(e)-\pi(u)+\pi(v)):\ e = uv\in E \text{ tight}\big).$$

Since x is a circulation the first sum is zero. Since x is feasible the second sum depends only on f, g, d, d' and π but not on the circulation x. This proves the lemma. ∎

An important consequence of the lemma is that if d and the identically zero function are (f, g)-equivalent, then every feasible circulation is d-minimal, that is, the d-minimizer of $P(f, g)$ is $P(f, g)$ itself.

By Lemma 2.1 to solve a minimum cost circulation problem it does not matter which of the (f, g)-equivalent costs are used, however, it will turn out that the one lying in the subspace

$$Q = \{x\in\mathbf{R}^E:\ x \text{ is a circulation and } x(e)=0 \text{ if } e \text{ is tight}\}$$

has particular advantages. Such a cost function is said to *fit* $P(f, g)$.

A version of the optimality criteria (Ford—Fulkerson [8]) states that if x is a feasible solution and π a potential such that

i) if $d(e)+\pi(u)-\pi(v)>0$ for $e=uv\in E$ then $x(e)=f(e)$

ii) if $d(e)+\pi(u)-\pi(v)<0$ for $|=uv\in E$ then $x(e)=g(e)$

then x is d-minimal and for any d-minimal circulation x' if $|d(e)+\pi(u)-\pi(v)|>0$ for an edge $e=uv\in E$ then $x'(e)=x(e)$.

Of course, if in i, we require $x(e)=f(e)$ only for edges $e=uv$ satisfying the stronger inequality $d(e)+\pi(u)-\pi(v)\geqq\alpha$ and in ii), $x(e)=g(e)$ is required only for edges satisfying $d(e)+\pi(u)-\pi(v)\leqq-\alpha$, where α is a positive number, then x is not necessarily a d-minimal circulation. However, if α is small relative to d, these weaker criteria can be interpreted so that x and π almost satisfy the optimality criteria and, then one may have an intuitive feeling that x is "almost" a d-minimal circulation. This feeling is justified somewhat by the following lemma.

Lemma 2.2. *Let x be a feasible circulation, d' a cost function, π a potential, and $\alpha>0$ a real number such that*

$$\textit{if}\quad d'(e)+\pi(u)-\pi(v)\geqq\alpha \quad \textit{for an edge}\quad e=uv \quad \textit{then}\quad x(e)=f(e),$$

$$\textit{if}\quad d'(e)+\pi(u)-\pi(v)\leqq-\alpha \quad \textit{for an edge}\quad e=uv \quad \textit{then}\quad x(e)=g(e).$$

Then for any d'-minimal circulation x'
$|d'(e)+\pi(u)-\pi(v)|\geqq|V|\alpha$ *for an edge* $e=uv$ *implies that* $x(e)=x'(e)$.

Proof. Without loss of generality we may assume that $x'\geqq x$: Otherwise reverse all edges $e\in E$ with $x'(e)<x(e)$. Replace $f(e)$, $g(e)$, $d'(e)$, $x(e)$ and $x'(e)$ by $-g(e)$, $-f(e)$, $-d'(e)$, $-x(e)$ and $-x'(e)$, respectively, on these edges. The new circulation problem with feasible circulation x also fulfils the conditions of the lemma. The new x' is a minimum cost circulation, and it satisfies the conclusions if and only if the original one does.

Now suppose indirectly that $|d'(e_0)+\pi(u_0)-\pi(v_0)|\geqq|V|\alpha$ and $x'(e_0)>x(e_0)$ for an edge $e_0=u_0v_0$. Since $d'(e)+\pi(u)-\pi(v)\leqq-\alpha$ would imply $x'(e)\geqq x(e)=g(e)\geqq x'(e)$, we have $d'(e_0)+\pi(u_0)-\pi(v_0)\geqq|V|\alpha$. Since $x'-x$ is a non-negative circulation there exists an oriented cycle C such that $e_0\in C$ and $x'-x$ is strictly positive on every edge of C. Let ε be the minimum value of $x'-x$ on C. Decrease the value of x' on the edges of C by ε and let x'' denote the new circulation. As $x'\geqq x''\geqq x$ the new circulation x'' is feasible. For every edge $e=uv$ of the cycle C we have $x(e)<x'(e)\leqq g(e)$, thus $d'(e)+\pi(u)-\pi(v)\geqq-\alpha$. The cost of the new circulation x'' is

$$\begin{aligned} d'x'' &= d'x'-\varepsilon\sum\big(d'(e)\colon e\in C\big) = d'x'-\varepsilon\sum\big(d'(e)+\pi(u)-\pi(v)\colon e=uv\in C\big)\\ &\leqq d'x'-\varepsilon\big(d'(e_0)+\pi(u_0)-\pi(v_0)-(|C|-1)\alpha\big)\\ &\leqq d'x'-\varepsilon\big(|V|\alpha-(|V|-1)\alpha\big) = d'x'-\varepsilon\alpha < d'x'. \end{aligned}$$

This contradicts the optimality of x'. ∎

It is important to emphasize that the classical Out-of-Kilter method (Ford—Fulkerson [8]) can be turned into a strongly polynomial algorithm if the costs are integers and do not exceed $\lceil |V|\sqrt{|E|}\rceil$ in absolute value. To this end one simply has to apply a strongly polynomial maximum flow computation in the flow augmenting phase of the Out-of-Kilter algorithm (Dinits [3], Edmonds—Karp [7]). When we refer to the Out-of-Kilter method we shall mean this version. It is well-known that if the costs are integers, there exists an optimal circulation and an integral potential satisfying the optimality criteria. The Out-of-Kilter method finds this integral potential.

Our algorithm will find the d-minimizer of $P(f, g)$. Having an optimal circulation x at hand, one can easily calculate a potential π satisfying the optimality criteria. Namely, one has to apply the Out-of-Kilter method with x as a starting feasible solution. Since x is an optimal solution, on the course of the algorithm no circulation change (breakthrough) may occur. Thus after at most $|E|^2$ potential changes a potential π is found such that the unchanged x and π satisfy the optimality criteria.

3. The algorithm

The core of the algorithm runs roughly as follows. First it is checked whether d is (f, g)-equivalent to the zero function. If so, then the d-minimizer of $P(f, g)$ is $P(f, g)$ itself and the algorithm is finished. If not, then the algorithm (by applying Gaussian elimination) determines the cost function which is (f, g)-equivalent to d and fits $P(f, g)$. Finally the Out-of-Kilter method is applied to a certain rounded (integer) cost function $\bar{d}$. This results in a circulation x and a potential π. With the help of lemma 2.2 we shall conclude that x is "good" on at least one non-tight edge e in the sence that $x(e)=x'(e)$ holds for any d-minimal circulation x'. Consequently, both the lower and the upper capacity of edge e can be changed to $x(e)$. This way, we get a new circulation polyhedron $P(f', g')$ such that

(a) The d-minimizer of $P(f, g)$ coincides with the d-minimizer of $P(f', g')$ and

(b) $T(f, g) \subset T(f', g')$ and $T(f, g) \neq T(f', g')$.

The edge e in question will easily be determined by using π.

Applying the above procedures at most $|E|$ times the algorithm finds the d-minimizer of $P(f, g)$.

Now we give the exact description of the algorithm.

The iterative step

Input *A digraph $D=(V, E)$ with upper and lower capacities and costs $f, g, d \in \mathbf{R}^E$, respectively, on the edges, such that there exists a feasible circulation.*

Output *Either d is (f, g)-equivalent to the identically zero function, or new capacities f' and g' are found which satisfy (a) and (b).*

0. *Let d^* be the projection of d to the subspace Q.*
If this projection is zero then d is (f, g)-equivalent to the identically zero function. STOP.
Otherwise let d' denote that positive multiple of d^* which has $d'_{\max} = |V|\sqrt{|E|}$.

1. *Define the following rounded cost function*

$$\bar{d}(e) = \lceil d'(e) \rceil.$$

*Use the Out-of-Kilter method to find a $\bar{d}$-minimal circulation x and a potential π such that x and π satisfy the optimality criteria (*i*) and* ii*)).*

2. *Define the new capacities as follows. Let $f'(e)=f(e)$ and $g'(e)=g(e)$ for edges $e=uv$ with $|d'(e)+\pi(u)-\pi(v)|<|V|$, and let $f'(e)=g'(e)=x(e)$ for edges $e=uv$ with $|d'(e)+\pi(u)-\pi(v)| \geqq |V|$.*

End

Now the whole algorithm is the following. Apply the iterative step repeatedly until getting capacities f^* and g^* such that d is (f^*, g^*)-equivalent to the identically zero function. Now the d-minimizer of $P(f, g)$ is $P(f^*, g^*)$.

Seemingly we use a new arithmetic operation, taking upper integer part. But, since we know the maximum absolute value of d', we can find $\bar{d}(e)$ for an edge e by $\log\left(\lceil |V| \sqrt{|E|} \rceil\right)$ comparisons.

Theorem 3.1. *The d-minimizer of $P(f, g)$ coincides with the d-minimizer of $P(f', g')$.*

Proof. Since $P(f, g) \supseteqq P(f', g')$ it suffices to show that $P_{\text{opt}} \subseteqq P(f', g')$. First observe that P_{opt} is the $\bar{d}'$-minimizer of $P(f, g)$, since the projection of d to Q results in a cost function d^* which is (f, g)-equivalent to d, and multiplying d^* by any positive number does not change the face optimizing it. Next we claim that the hypotheses of Lemma 2.2 hold for d', x, π and $\alpha = 1$. Indeed, let $e = uv \in E$ and $d'(e) + \pi(u) - \pi(v) \leqq -1$. Then ▌

$$\bar{d}(e) + \pi(u) - \pi(v) = \left(d'(e) + \pi(u) - \pi(v)\right) + \left(\bar{d}(e) - d'(e)\right) < -1 + 1 = 0$$

by the definition of $\bar{d}$. Thus the optimality criteria for x and π imply that $x(e) = g(e)$.

It can be seen similarly that $d'(e) + \pi(u) - \pi(v) \geqq 1$ implies $x(e) = f(e)$. Now Lemma 2.2 proves the theorem. ▌

Theorem 3.2. $T(f, g) \supset T(f', g')$ *and* $T(f, g) \neq T(f', g')$.

Proof. Since $T(f, g) \supseteqq T(f', g')$ we only have to prove that $|d'(e) + \pi(u) - \pi(v)| \geqq |V|$ for at least one non-tight edge $e = uv$. Let us define the following vector d''

$$d''(e) = \begin{cases} d'(e) + \pi(u) - \pi(v) & \text{if } e = uv \in E \text{ is not tight} \\ 0 & \text{othervise.} \end{cases}$$

What we have to prove is now $d''_{\max} \geqq |V|$. Since d' fits $P(f, g)$ the vectors d' and $d'' - d'$ are orthogonal and so $|d'| \leqq |d''|$. Thus we have

$$d''_{\max} \geqq \left(1/\sqrt{|E|}\right)|d''| \geqq \left(1/\sqrt{|E|}\right)|d'| \geqq \left(1/\sqrt{|E|}\right)d'_{\max} = \lceil |V| \sqrt{|E|} \rceil / \sqrt{|E|} \geqq |V|.$$

Here the last equality is due to the definition of d'. ▌

4. Complexity

First we survey the complexity of the algorithms previously known. The first polynomial algorithm for the maximum flow problem was given by Dinits [3] and Edmonds and Karp [7]. The Out-of-Kilter method for minimum cost flow problems was introduced by Fulkerson [9] and Minty [13]. If the capacities are integers this algorithm has running time $O\left(|E|^3 \cdot C\right)$, where C is the maximum capacity. In the case when the costs are integers (but not necessarily the capacities) the algorithm can be considered as $O\left(|V| \cdot D\right)$ maximum flow computations, where D is the maximum absolute value of the costs (Edmonds—Karp [7]).

The first polynomial time algorithm for this problem was given by Edmonds and Karp [7]. They developed a scaling technique to solve minimum cost flow problems with integer capacities. They algorithm runs in $O(|E|^4 \log C)$ time. Scaling the costs instead of the capacities (if the costs are integers, while the capacities may be arbitrary reals) one can obtain a minimum cost flow algorithm, which consists of $O(|E| \log D)$ maximum flow computations (Röck [14]).

Now we turn to computing the complexity of our algorithm. By Theorem 3.2, to solve the minimum cost circulation problem one has to apply the iterative step at most $|E|$ times.

First we describe the computational details of step 0, projecting the cost function d to the subspace

$$Q = \{x \in \mathbf{R}^E\colon x \text{ is a circulation and } x(e)=0 \text{ if } e \text{ is tight}\}.$$

Observe that $d^*(e)=0$ for all tight edges e. Define the node—edge incidence matrix of the graph $(V, E-T(f,g))$ by

$$A(v,e) = \begin{cases} 1 & \text{if } e = uv \\ -1 & \text{if } e = vu \\ 0 & \text{otherwise.} \end{cases}$$

The vector d^* restricted to the coordinates $E-T(f,g)$ is orthogonal to the rows of A and is of the form $d|(E-T(f,g))+\pi A$ for a suitable potential π. The required π can be computed by solving the equality system

$$(d|(E-T(f,g))+\pi A)A^t = 0.$$

This is a linear equality system in $|V|$ variables. Solving it takes $O(|V|^3)$ steps by Gaussian elimination. The Gaussian elimination is known to be a strongly polynomial algorithm (Edmonds [4]), that is, the size of the numbers occuring during the elimination is polynomially bounded in the size of the input numbers. Finding the integral capacities $\bar{d}$ takes $O(|E| \log |E|)$ time.

Next we deal with the time necessary to solve the minimum cost circulation problem in the iterative step. The cost function $\bar{d}$ is integral and $\bar{d}_{\max} \leq |V| \sqrt{|E|}$. Thus the Out-of-Kilter finds a solution via $O(|V|^2 \sqrt{|E|})$ maximum flow computations.

Finally, after having computed a d-minimal circulation x the computation of a potential π such that x and π satisfy the optimality criteria takes only $O(|E|^3)$ time.

To see the overall running time of the algorithm, let T denote the running time of the maximum flow algorithm used. The solution of a minimum cost circulation problem takes

$$O(|E|(|V|^3 + |E| \log(|E|^{1/2}|V|) + |V|^2|E|^{1/2}T) + |E|^3) = O(|E|^{3/2}|V|^2 T)$$

time.

Remark. One can speed up the algorithm by using the cost scaling technique in the computation of a $\bar{d}$-minimal circulation. This improves the running time to $O(|E|^2 T \log |E|)$.

5. Extensions to more general problems

First we discuss the circulation problem with possibly infinite capacities. A slight modification of the algorithm works also in this more general case.

Observe that we can reduce the circulation problem to one with non-negative lower capacities: Reverse all edges e with $f(e) \leqq g(e) < 0$ and put $-g(e)$, $-f(e)$ and $-d(e)$ as capacities and costs. Add an edge in the reverse direction for edges e with $f(e) < 0 < g(e)$ and put $0, g(e), d(e)$ and $0, -f(e), -d(e)$ as capacities and costs on the edge and its reverse copy respectively.

The algorithm in section 3 applies to this reduced problem without any change. The only apparent danger is that in the iterative step no $\bar{d}$-minimal circulation exists (that is, the minimum is minus infinite). In this case the same is true in the original problem, because $\bar{d} \geqq d'$ and $g \geqq f \geqq 0$.

The idea of the present algorithm can be applied to the general linear programming problem. This will be discussed in a forthcoming paper. There we give an algorithm which solves the linear programming problem

$$A \cdot x = b$$

$$x \geqq 0$$

$$\min c \cdot x$$

in time polynomial in the size of A (that is, the number of arithmetic steps is independent of the size of the vectors b and c). It makes use of any polynomial-time linear programming algorithm.

The first version of this algorithm was presented at the Colloquium on Algorithms and Complexity, 1984 Pécs, Hungary. That algorithm used the Edmonds—Karp scaling technique and Diophantine approximation. The later was made possible by Lovász's lattice reduction algorithm (Lenstra—Lenstra—Lovász [10]). After the Colloquium I have found the easier and quicker algorithm presented here. The original algorithm has the advantage that its solution idea can be applied to solve submodular flow problems (Edmonds—Giles [6]) in strongly polynomial time. The submodular flow problem includes besides the minimum cost circulation problem, many other combinatorial problems, such as the directed cut covering problem (Lucchesi—Younger [11]) and the polymatroid intersection problem (Edmonds [5]). Cunningham and Frank [2] gave a combinatorial polynomial algorithm for solving this problem. The strongly polynomial algorithm will be presented in a forthcoming paper by A. Frank and the present author.

Acknowledgements. I wish to express my thanks to András Frank for suggesting this problem to me and for many stimulating discussions on the algorithm presented here. Thanks are due to András Sebő for improving the presentation of this paper.

References

[1] R. P. Anstee, A polynomial algorithm for b-matching: an alternative approach, *Research Report CORR 83—22*, University Waterloo, Waterloo, Ontario, 1983.
[2] W. H. Cunningham and A. Frank, A primal dual algorithm for submodular flows; *Mathematics of Operations Research*, **10** (1985), 251—262.
[3] E. A. Dinits, Algorithm for solution of a problem of maximum flow in a network with power estimation; *Soviet Math. Dokl.*, **11** (1970), 1277—1280.
[4] J. Edmonds, Systems of distinct representatives and linear algebra; *J. Res. Nat. Bur. Standards*, **71***B* (1967), 241—245.
[5] J. Edmonds, Submodular functions, matroids and certain polyhedra, *in: Combinatorial Structures and Applications*, (R. K. Guy et al. eds.) 1970 Gordon and Breach, New York 67—87.
[6] J. Edmonds and R. Giles, A min-max relation for submodular functions on graphs, *Annals of Discrete Math.* **1** (1977), 185—204.
[7] J. Edmonds and R. M. Karp, Theoretical improvements in algorithmic efficiency for network flow problems, *J. ACM* **19** (1972), 248—264.
[8] L. R. Ford, jr. and D. R. Fulkerson, *Flows in Networks*, Princeton University Press, Princeton N. J., 1962.
[9] D. R. Fulkerson, An Out-of-Kilter method for minimal cost flow problems, *J. Soc. Indust. Appl. Math.* **9** (1961), 18—27.
[10] A. K. Lenstra, H. W. Lenstra, jr. and L. Lovász, Factoring polynomials with rational coefficients, *Math. Ann.* **261** (1982), 515—534.
[11] C. L. Lucchesi and D. H. Younger, A minimax theorem for directed graphs, *J. London Math. Soc.* **17** (1978), 368—374.
[12] N. Megiddo, Towards a genuinely polynomial algorithm for linear programming, *SIAM J. Comput.* **12** (1981), 347—353.
[13] G. J. Minty, Monotone Networks, *Proc. Roy. Soc. London, Ser. A*, **257** (1960), 194—212.
[14] H. Röck, Scaling techniques for minimal cost network flows, *in: Discrete Structures and Algorithms*, (U. Page ed.) 1980, Carl Hanser, München, 181—191.

Éva Tardos

Research Institute for Telecommunication
1026 Budapest, Gábor Á. u. 65.
Hungary

Present address:
Institut für Ökonometrie und Op. Res.
Universitat Bonn
Bonn, West Germany